Modified Expansion Theorem for Sturm-Liouville problem with transmission conditions

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Abstract : This paper is devoted to the derivation of expansion a associated with a discontinuous Sturm-Liouville problems defined on $[-\pi, 0) \cup (0, \pi]$. We derive an eigenfunction expansion theorem for the Green's function of the problem as well as a theorem of uniform convergence of a certain class of functions.

Keywords : Sturm-Liouville problems, transmission conditions, expansions theorem, Carleman equation.

1 Introduction

The importance of Sturm-Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a differential equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm-Liouville problem. The issue of expansion in generalized eigenfunctions is a classical one going back at least to Fourier. A relatively recent impact is due to the study of wave propagation in random media [1, 6], where eigenfunction expansions are an important input in the proof of localization. The use of this tool is settled by classical results in the Schrdinger operator case. But with the study of operators related with classical waves, [1, 7], a need for more general results on eigenfunction expansion became apparent. Eigenfunction expansions problems for Sturm-Liouville problems have been investigated by many authors, see [2, 8]

In this paper we shall investigate one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$\Gamma(y) := -y''(x, \lambda) + q(x)y(x, \lambda) = \lambda y(x, \lambda) \tag{1.1}$$

to hold in finite interval $(-\pi, \pi)$ except at one inner point $0 \in (-\pi, \pi)$,

where discontinuity in u and u' are prescribed by transmission conditions

$$\Gamma_1(y) := a_1 y'(0-,\lambda) + a_2 y(0-,\lambda) + a_3 y'(0+,\lambda) + a_4 y(0+,\lambda) = 0, \quad (1.2)$$

$$\Gamma_2(y) := b_1 y'(0-,\lambda) + b_2 y(0-,\lambda) + b_3 y'(0+,\lambda) + b_4 y(0+,\lambda) = 0, \quad (1.3)$$

together with the boundary conditions

$$\Gamma_3(y) := \cos\alpha y(-\pi,\lambda) + \sin\alpha y'(-\pi,\lambda) = 0, \quad (1.4)$$

$$\Gamma_4(y) := \cos\beta y(\pi,\lambda) + \sin\beta y'(\pi,\lambda) = 0. \quad (1.5)$$

Such problems with point interactions are also studied in [4], etc. Boundary value problems which may have discontinuities in the solution or its derivative at an interior point are also studied. Conditions are imposed on the left and right limits of solutions and their derivatives at an interior point and are often called transmission conditions" or interface conditions". These problems often arise in varied assortment of physical transfer problems, see [3, 5]. Also, some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness).

It is the purpose of this paper to present a new and somewhat extensive family of orthogonal expansions associated with an discontinuous Sturm-liouville problem. By using an own technique we introduce a new equivalent inner product in the Hilbert space $L_2(-\pi,0) \oplus L_2(0,\pi)$ and a linear operator in it such a way that the considered problem can be interpreted as eigenvalue problem for this operator. Using a new approach to the now classical theory of Sturm-Liouville eigenfunction expansions, based essentially on the method of integral equations. Moreover, we introduce a expansion method for solution of carlemans equation.

2 Preliminary Results

Let T=$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$. Denote the determinant of the i-th and j-th columns of the matrix T by ρ_{ij}. Note that throughout this study we shall assume that $\rho_{12} > 0$ and $\rho_{34} > 0$.

In this section we shall define two basic solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ by own technique as follows. At first, let us consider solutions of the equation (1.1) on the left-hand $[-\pi,0)$ of the considered interval $[\pi,0)\cup(0,\pi]$ satisfying the initial conditions

$$y(-\pi,\lambda) = \sin\alpha, \quad \frac{\partial y(-\pi,\lambda)}{\partial x} = -\cos\alpha \quad (2.1)$$

By virtue of well-known existence and uniqueness theorem of ordinary differential equation theory this initial-value problem for each λ has a unique solution $\phi_1(x,\lambda)$. Moreover [[8], Teorem 7] this solution is an entire function of λ for each fixed $x \in [-\pi, 0)$. Using this solutions we can prove that the equation (1.1) on the right-hand interval $\in (0,\pi]$ of the considered interval $[-\pi,0) \cup (0,\pi]$ has the solution $u = \phi_2(x,\lambda)$ satisfying the initial conditions

$$y(0,\lambda) = \frac{1}{\rho_{12}}(\rho_{23}\phi_1(0,\lambda) + \rho_{24}\frac{\partial\phi_1(0,\lambda)}{\partial x}) \tag{2.2}$$

$$y'(0,\lambda) = \frac{-1}{\rho_{12}}(\rho_{13}\phi_1(0,\lambda) + \rho_{14}\frac{\partial\phi_1(0,\lambda)}{\partial x}). \tag{2.3}$$

By applying the method of [4] we can prove that the equation (1.1) on $(0,\pi]$ has an unique solution $\phi_2(x,\lambda)$ satisfying the conditions (2.2)-(2.3) which also is an entire function of the parameter λ for each fixed $x \in (0,\pi]$. Consequently, the function $\phi(x,\lambda)$ defined by

$$\phi(x,\lambda) = \{ \begin{array}{l} \phi_1(x,\lambda) \quad \text{for } x \in [-\pi,0) \\ \phi_2(x,\lambda) \quad \text{for } x \in (0,\pi]. \end{array} \tag{2.4}$$

satisfies equation (1.1), the first boundary condition (1.4) and the both transmission conditions (1.2) and (1.3). Similarly, $\chi_2(x,\lambda)$ be solutions of equation (1.1) on the left-right interval $(0,\pi]$ subject to initial conditions

$$y(\pi,\lambda) = -\sin\beta, \quad \frac{\partial y(\pi,\lambda)}{\partial x} = \cos\beta. \tag{2.5}$$

By virtue of [[8], Teorem 7] each of these solutions are entire functions of λ for fixed x. By applying the same technique we can prove there is an unique solution $\chi_1(x,\lambda)$ of equation (1.1) the left-hand interval $[-\pi,0)$ satisfying the initial condition

$$y(0,\lambda) = \frac{-1}{\rho_{34}}(\rho_{14}\chi_2(0,\lambda) + \rho_{24}\frac{\partial\chi_2(0,\lambda)}{\partial x}), \tag{2.6}$$

$$y'(0,\lambda) = \frac{1}{\rho_{34}}(\rho_{13}\chi_2 + \Delta_{23}\frac{\partial\chi_2(0,\lambda)}{\partial x}). \tag{2.7}$$

By applying the similar technique as in [4] we can prove that the solutions $\chi_1(x,\lambda)$ are also an entire functions of parameter λ for each fixed x. Consequently, the function $\chi(x,\lambda)$ defined as

$$\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda), & x \in [-\pi,0) \\ \chi_2(x,\lambda), & x \in (0,\pi] \end{cases}$$

satisfies the equation (1.1) on whole $[-\pi, 0) \cup (0, \pi]$, the other boundary condition (1.5) and the both transmission conditions (1.2) and (1.3). In the Hilbert Space $\mathcal{H} = L_2[-\pi, 0) \oplus L_2(0, \pi]$ of two-component vectors we define an inner product by

$$< y, z >_{\mathcal{H}} := \rho_{12} \int_{-\pi}^{0} y(x)\overline{z(x)}dx + \rho_{34} \int_{0}^{\pi} y(x)\overline{z(x)}dx$$

for $y = y(x),\ z = z(x)\ \in \mathcal{H}$. We introduce the linear operator $A : \mathcal{H} \to \mathcal{H}$ with domain of definition satisfying the following conditions
i) y and y' are absolutely continuous in each of intervals $[-\pi, 0)$ and $(0, \pi]$ and has a finite limits $y(c\mp)$ and $y'(c\mp)$
ii) $\Gamma y(x) \in \mathcal{H},\ \Gamma_1 y(x) = \Gamma_2 y(x) = \Gamma_3 y(x) = \Gamma_4 y(x) = 0,$
Obviously $D(A)$ is a linear subset dense in $\mathcal{H}$. We put

$$(Ay)(x) = \Gamma y(x),\ x \in \mathcal{H}$$

for $y \in D(A)$. Then the problem $(1.1) - (1.5)$ is equivalent to the equation

$$Ay = \lambda y$$

in the Hilbert space $\mathcal{H}$. Taking in view that the Wronskians $W(\phi_i, \chi_i; x) := \phi_i(x, \lambda)\chi_i'(x, \lambda) - \phi_i'(x, \lambda)\chi_i(x, \lambda)$ are independent of variable x we shall denote $w_i(\lambda) = W(\phi_i, \chi_i; x)$ $(i = 1, 2)$. By using (1.2) and (1.3) we have $\rho_{12} w_1(\lambda) = \rho_{34} w_2(\lambda)$ for each $\lambda \in \mathbb{C}$. It is convenient to introduce the notation

$$w(\lambda) := \rho_{34} w_1(\lambda) = \rho_{12} w_2(\lambda). \tag{2.8}$$

Theorem 2.1. *For all $y, z \in D(A)$, the equality*

$$< Ay, z > = < y, Az > \tag{2.9}$$

holds.

Proof. Integrating by parts we have for all $y, z \in D(A)$,

$$\begin{aligned} < Ay, z > &= \rho_{12} \int_{-\pi}^{0} \Gamma y(x)\overline{z(x)}dx + \rho_{34} \int_{0}^{\pi} \Gamma y(x)\overline{z(x)}dx \\ &= \rho_{12} \int_{-\pi}^{0} y(x)\overline{\Gamma z(x)}dx + \rho_{34} \int_{0}^{\pi} y(x)\overline{\Gamma z(x)}dx + \rho_{12} W[y, \overline{z}; 0-] \\ &- \rho_{12} W[y, \overline{z}; -\pi] + \rho_{34} W[y, \overline{z}; \pi] - \rho_{34}\ W[y, \overline{z}; 0] \\ &= < y, Az > + \rho_{12} W[y_0, \overline{z}; 0] - \rho_{12} W[y, \overline{z}; -\pi] \\ &+ \rho_{34} W[y, \overline{z}; \pi] - \rho_{34}\ W[y, \overline{z}; 0] \end{aligned} \tag{2.10}$$

From the boundary conditions (1.2)-(1.3) it is follows obviously that

$$W(y,\overline{z};-\pi)=0 \text{ and } W(y,\overline{z};\pi)=0 \tag{2.11}$$

The direct calculation gives

$$\rho_{12}W(y,\overline{z};0)=\rho_{34}W(y,\overline{z};0). \tag{2.12}$$

Substituting (2.11) and (2.12) in (2.10) we obtain the equality (2.9). □

Relation (2.9) shows that the operator A is symmetric and selfa-adjoint. Therefore all eigenvalues of the operator A are real and two eigenfunctions corresponding to the distinct eigenvalues are orthogonal in the sense of the following equality

$$\rho_{12}\int\limits_{-\pi}^{0} y(x)z(x)dx+\rho_{34}\int\limits_{0}^{\pi} y(x)z(x)dx=0. \tag{2.13}$$

3 Expansion theorem by Green function's method

To present its explicit form we introduce the Green function

$$G(x,s;\lambda)=\begin{cases}\frac{\phi(x,\lambda)\chi(s,\lambda)}{\omega(\lambda)} & \text{for } -\pi\le s\le x\le\pi,\quad x,s\neq 0\\[2ex] \frac{\phi(s,\lambda)\chi(x,\lambda)}{\omega(\lambda)} & \text{for } -\pi\le x\le s\le\pi,\quad x,s\neq 0\end{cases} \tag{3.1}$$

for $x,s\in[-\pi,0)\cup(0,\pi]$ where $\phi(x,\lambda)$ and $\chi(x,\lambda)$ are solutions of the boundary-value- transmission problem (1.1)-(1.5). It symmetric with respect to x and s, and real-valued for real λ. We show that the function

$$y(x,\lambda) \;=\; \rho_{12}\int\limits_{-\pi}^{0} G(x,s;\lambda)f(s)ds+\rho_{34}\int\limits_{0}^{\pi} G(x,s;\lambda)f(s)ds \tag{3.2}$$

called a resolvent is a solution of the equation

$$y''+\{\lambda-q(x)\}y=f(x), \tag{3.3}$$

(where $f(x) \neq 0$ is a continuous function), satisfying the boundary-transmission conditions (1.2)-(1.5). We can assume that $\lambda = 0$ is not an eigenvalue. Otherwise, we take a fixed number η, and consider the boundary-value-transmission problem

$$y''(x,\lambda) + \{(\lambda+\eta) - q(x)\}y(x,\lambda) = 0 \tag{3.4}$$

$$a_1 y'(0-,\lambda) + a_2 y(0-,\lambda) + a_3 y'(0+,\lambda) + a_4 y(0+,\lambda) = 0, \tag{3.5}$$

$$b_1 y'(0-,\lambda) + b_2 y(0-,\lambda) + b_3 y'(0+,\lambda) + b_4 y(0+,\lambda) = 0, \tag{3.6}$$

$$\cos\alpha y(-\pi,\lambda) + \sin\alpha y'(-\pi,\lambda) = 0, \tag{3.7}$$

$$\cos\beta y(\pi,\lambda) + \sin\beta y'(\pi,\lambda) = 0 \tag{3.8}$$

with the same eigenfunction as for the problem (1.1)-(1.4). All the eigenvalues are shifted through η to the right. It is evident that η can be selected so that 0 is not an eigenvalue of the new problem.

Let $G(x,s;0) = G(x,s)$ then the function

$$y(x,\lambda) \;=\; \rho_{12}\int_{-\pi}^{0} G(x,s)f(s)ds + \rho_{34}\int_{0}^{\pi} G(x,s)f(s)ds \tag{3.9}$$

is a solution of the equation $y'' - q(x)y = f(x)$, and satisfies the boundary-transmission conditions (1.2)-(1.4). We rewrite (3.3) in the form

$$y'' - q(x)y = f(x) - \lambda y \tag{3.10}$$

Thus, the homogeneous problem ($f(x) \equiv 0$) is equivalent to the integral equation

$$y(x,\lambda) + \lambda\{\rho_{12}\int_{-\pi}^{0} G(x,s)y(s)ds + \rho_{34}\int_{0}^{\pi} G(x,s)y(s)ds\} = 0 \tag{3.11}$$

Denote by $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_n, ...$ the collection of all the eigenvalues of the problem (1.1)-(1.4), and the corresponding normalized eigenfunctions by $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_n, ...$

$$K(x,\xi) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}.$$

By the asymptotic formulas for the eigenvalues, obtained in previous section, the series for $H(x,\xi)$ converges absolutely and uniformly; therefore, $K(x,\xi)$ is continuous. Consider the kernel

$$P(x,\xi) = G(x,\xi) + K(x,\xi) = G(x,\xi) + \sum_{n=0}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}$$

which is obviously continuous and symmetric. By familiar theorem in the theory of integral equations, any symmetric kernel $P(x,\xi)$ which is not identically zero has at least one eigenfunction [**?**], i.e., there is a number λ_0 and a function $u(x) \neq 0$ satisfying the equation

$$u(x,\lambda) + \lambda_0\{\rho_{12} \int_{-\pi}^{0} P(x,\xi)u(\xi)d\xi + \rho_{34} \int_{0}^{\pi} P(x,\xi)u(\xi)d\xi\} = 0. \quad (3.12)$$

Thus, if we show that the kernel has no eigenfunctions, we obtain $P(x,\xi) \equiv 0$, i.e.,

$$G(x,\xi) = -\sum_{n=0}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}. \quad (3.13)$$

Hence, to obtain the completeness of eigenfunctions is now easy. It follows from the equation (3.11) that

$$\rho_{12} \int_{-\pi}^{0} G(x,\xi)\varphi(\xi)d\xi + \rho_{34} \int_{0}^{\pi} G(x,\xi)\varphi(\xi)d\xi = -\frac{1}{\lambda_n}\varphi_n(x) \quad (3.14)$$

therefore,

$$\rho_{12} \int_{-\pi}^{0} P(x,\xi)\varphi(\xi)d\xi + \rho_{34} \int_{0}^{\pi} P(x,\xi)\varphi(\xi)d\xi = 0 \quad (3.15)$$

i.e., the kernel $Q(x,\xi)$ is orthogonal to all eigenfunctions of the boundary-value-transmission problem (1.1)-(1.4). Let $u(x)$ be a solution of the integral equation (3.12). We show that $u(x)$ is orthogonal to all $\varphi_n(x)$. In fact, it follows from (3.12)

$$0 = \rho_{12} \int_{-\pi}^{0} u(x)\varphi_n(x)dx + \rho_{34} \int_{0}^{\pi} u(x)\varphi_n(x)$$

therefore,

$$\begin{aligned} 0 &= u(x,\lambda) + \lambda_0\{\rho_{12}\int_{-\pi}^{0} P(x,\xi)u(\xi)d\xi + \rho_{34}\int_{0}^{\pi} P(x,\xi)u(\xi)d\xi\} \\ &= u(x,\lambda) + \lambda_0\{\rho_{12}\int_{-\pi}^{0} G(x,\xi)u(\xi)d\xi + \rho_{34}\int_{0}^{\pi} G(x,\xi)u(\xi)d\xi\}, \end{aligned}$$

i.e., $u(x,\lambda)$ is an eigenfunction of the boundary-value-transmission problem (1.1)-(1.4). Since it is orthogonal to all $\varphi_n(x)$, it is also orthogonal to itself, with the consequence that $u(x,\lambda) = 0$ and $P(x,\xi) = 0$. The formula (3.13) is thus proved.

Theorem 3.1. *(Expansion Theorem) If $f(x)$ has a continuous second derivative and satisfies the boundary-transmission conditions (1.2)-(1.4), then $f(x)$ can be expanded into an absolutely and uniformly convergent Fourier series of eigenfunctions of the boundary-value-transmission problem (1.1)-(1.4) on $[-1,0) \cup (0,1]$, i.e.,*

$$f(x) = \sum_{n=0}^{\infty} c_n\varphi_n(x) \tag{3.16}$$

where c_n are the Fourier coefficients of f given by

$$c_n = \rho_{12}\int_{-\pi}^{0} f(x)\varphi_n(x)dx + \rho_{34}\int_{0}^{\pi} f(x)\varphi_n(x)dx. \tag{3.17}$$

Proof. □

Theorem 3.2. *(Parseval equality) For any square-integrable function $f(x)$ in the interval $[-1,0) \cup (0,1]$, the Parseval equality*

$$\rho_{12}\int_{-\pi}^{0} f^2(x)dx + \rho_{34}\int_{0}^{\pi} f^2(x)dx = \sum_{n=0}^{\infty} c_n^2 \tag{3.18}$$

holds.

Proof. □

4 The modified Carleman equation

We now return to the formula

$$y(x,\lambda) \;=\; \rho_{12}\int_{-\pi}^{0} G(x,s;\lambda)f(s)ds + \rho_{34}\int_{0}^{\pi} G(x,s;\lambda)f(s)ds \quad (4.1)$$

whose right-hand side has been called the resolvent. Let

$$y(x,\lambda) = \sum_{n=0}^{\infty} b_n(\lambda)\varphi_n(x),\; c_n = \rho_{12}\int_{-\pi}^{0} f(x)\varphi_n(x)dx + \rho_{34}\int_{0}^{\pi} f(x)\varphi_n(x)dx \quad (4.2)$$

Then, we have

$$\begin{aligned} c_n &= \rho_{12}\int_{-\pi}^{0}\{y(x)'' - q(x)y(x\}\varphi_n(x)dx + \rho_{34}\int_{0}^{\pi}\{y(x)'' - q(x)y(x)\}\varphi_n(x)dx \\ &= -\lambda_n b_n(\lambda) + b_n(\lambda). \end{aligned} \quad (4.3)$$

Hence, $b_n(\lambda) = \frac{c_n}{\lambda-\lambda_n}$ and the expansion of the resolvent is

$$y(x,\lambda) = \rho_{12}\int_{-\pi}^{0} G(x,s;\lambda)f(s)ds + \rho_{34}\int_{0}^{\pi} G(x,s;\lambda)f(s)ds = \sum_{n=0}^{\infty}\frac{c_n}{\lambda-\lambda_n} \quad (4.4)$$

An important formula can now be derived from the above. Substituting the equality

$$c_n = \rho_{12}\int_{-\pi}^{0} f(s)\varphi_n(s)ds + \rho_{34}\int_{0}^{\pi} f(s)\varphi_n(s)ds \quad (4.5)$$

on the right-hand side, we see that

$$\begin{aligned} &\rho_{12}\int_{-\pi}^{0} G(x,s;\lambda)f(s)ds + \rho_{34}\int_{0}^{\pi} G(x,s;\lambda)f(s)ds \\ &= \sum_{n=0}^{\infty}\frac{\varphi_n(x)}{\lambda-\lambda_n}\{\rho_{12}\int_{-\pi}^{0} f(s)\varphi_n(s)dx + \rho_{34}\int_{0}^{\pi} f(s)\varphi_n(s)dx\}. \end{aligned} \quad (4.6)$$

Since f(s) is arbitrary,

$$G(x,s;t)=\sum_{n=0}^{\infty}\frac{\varphi_n(x)\varphi_n(s)}{t-\lambda_n}. \tag{4.7}$$

Thus we obtain

$$\rho_{12}\int_{-\pi}^{0}G(x,x;t)dt+\rho_{34}\int_{0}^{\pi}G(x,x;t)dx=\sum_{n=0}^{\infty}\frac{1}{t-\lambda_n} \tag{4.8}$$

Put $N(\lambda)=\sum_{0\leq\lambda_n\leq\lambda}1$ is number of eigenvalues λ_n less than λ. we get from t (4.8) the modified Carleman equation

$$\rho_{12}\int_{-\pi}^{0}G(x,x;t)dx+\rho_{34}\int_{0}^{\pi}G(x,x;t)dx=\sum_{n=0}^{\infty}\frac{dN(\lambda)}{t-\lambda}. \tag{4.9}$$

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